CALCULATION OF THE MAGNETIC FIELD OF A COIL

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An approximate method is given for calculating the magnetic field of a coil of circular cross section with continuous coil windings.

At the present time much attention is being devoted to the study of electromagnetic heating and drying of materials by industrial-frequency currents. Briefly this new method of thermal preparation of materials consists in the following. The material, together with ferromagnetic elements, is placed in the alternating electromagnetic field of a coil, as a result of which heat is generated in the ferromagnetic elements and is transferred immediately to the material. The magnetic field strength at different points in space must be known to determine the heat sources. Existing formulas for calculating magnetic fields are either too complicated for practical use, or enable the field to be calculated only on the axis of the coil [1].

It is thus necessary to develop an approximate method for calculating the magnetic fields of coils of various transverse cross sections, i.e., one which is also convenient for taking into account magnetic materials distributed inside the coil.

The present article considers the simplest case in which the magnetic field is calculated for a coil of circular transverse cross section with continuous coil windings but with no magnetic materials on the inside.

It should be noted that this method is also suitable for making calculations for coils of other transverse cross sections when magnetic materials are present inside the coils, although the calculations in this case are much more complicated.

The mathematical problem is formulated in the following manner.

It is well known that the magnetic scalar potential φ satisfies Laplace's equation at all points in space where there is no electric current:

$$\Delta\varphi = 0, \tag{1}$$

where Δ is the Laplacian operator.

The magnetic field strength H is determined from the scalar potential as follows:

$$\mathbf{H} = -\operatorname{grad}\varphi. \tag{2}$$

In view of the angular symmetry of the problem in our case ($\varphi = \varphi(r, z)$, $\partial\varphi/\partial\vartheta = 0$), Eq. (1) is written in cylindrical coordinates as

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial\varphi}{\partial r} + \frac{\partial^2\varphi}{\partial z^2} = 0. \tag{3}$$

The problem in this case is therefore two-dimensional. We now write the boundary conditions for Eq. (3) (see the figure).

The condition of symmetry relative to the z-axis (absence of the components $H_r = -\partial\varphi/\partial r$ at points on the z-axis) is written as

$$\left.\frac{\partial\varphi(r, z)}{\partial r}\right|_{r=0} = 0 \quad (-\infty < z < \infty). \tag{4}$$

The following condition is also obvious:

$$\varphi(r, z)_{r,z\to\infty} \to 0. \tag{5}$$

It is well known that the normal component of the magnetic field strength is continuous when crossing the current surface, and that the tangential component is discontinuous by an amount equal to the number A of amp-turns. We thus write the conditions at the surface of the coil in the form

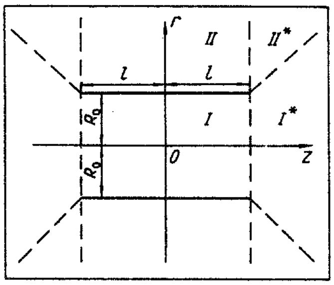

Diagram of the axial cross section of the coil.

$$\left[\frac{\partial\varphi_{\rm I}(r, z)}{\partial r} - \frac{\partial\varphi_{\rm II}(r, z)}{\partial r}\right]_{r=\pm R_0} = 0, \tag{6}$$

$$\left[\frac{\partial\varphi_{\rm I}(r, z)}{\partial z} - \frac{\partial\varphi_{\rm II}(r, z)}{\partial z}\right]_{r=\pm R_0} = A \tag{7}$$

$$(-l < z < l).$$

Since the scalar magnetic potential is a multivalued function, we must impose one more condition on the function $\varphi(r, z)$ itself. On completing a single circuit of a field line we have

$$\varphi_{\rm I}(r, z)|_{z=+0} - \varphi_{\rm I}(r, z)|_{z=-0} = 2lA. \tag{8}$$

We seek a solution of Eq. (3) with the boundary conditions (4)–(8) by reducing it to ordinary differential equations [2].

Since the problem is symmetric it is sufficient to find the solution in the first quadrant of the (z, r) plane. The quadrant is divided into four parts (see the figure), and the line dividing regions II and II* has the equation $R = z - \gamma$, where $\gamma = l - R_0$.

Keeping the distribution of the dipole magnetic fields in mind, we find it convenient to seek a solution of the problem in the form of polynomials in some degree of the variable r.

We represent the solutions in the appropriate regions in the form

$$\varphi_{I}(r, z)=\sum_{n=0}^{N} a_n(z)\left(\frac{r}{R_0}\right)^{2n},$$

$$\varphi_{I}^{*}(r, z)=\sum_{n=0}^{N} a_n^{*}(z)\left(\frac{r}{R(z)}\right)^{2n},$$

$$\varphi_{II}(r, z)=\sum_{n=1}^{M} b_n(z)\left(\frac{R_0}{r}\right)^{2n+1},$$

$$\varphi_{II}^{*}(r, z)=\sum_{n=1}^{M} b_n^{*}(z)\left(\frac{R(z)}{r}\right)^{2n+1}. \qquad (9)$$

We impose the requirement that the expression $\Delta\varphi_{I}(r,z)$ should be orthogonal with respect to the functions r^{2k-2} $(k = 1, 2, \ldots, N)$ with weight r over the segment $(0, R_0)$, and that the expression $\Delta\varphi_{II}(r, z)$ should be orthogonal relative to the functions $r^{-(2k+1)}$ $(k = 1, 2, \ldots, M - 1)$ with weight r^3 over the segment (R_0, ∞). We thus obtain

$$\int_0^{R_0} \Delta\varphi_{I}(r, z) r^{2k-1} dr = 0 \quad (k = 1, 2, \ldots, N), \qquad (10)$$

$$\int_{R_0}^{\infty} \Delta\varphi_{II}(r, z) r^{-2(k-1)} dr = 0$$

$$(k = 1, 2, \ldots, M-1). \qquad (11)$$

Expressions (10) and (11) give us a system of $(N + M - 1)$ ordinary second-order differential equations for the $(N + M + 1)$ unknown functions $\{a_n(z)\}$, $\{b_m(z)\}$ $(n = 1, 2, \ldots, N;\ m = 1, 2, \ldots, M)$. The two equations which are lacking may be obtained from the boundary conditions (6) and (7).

We write this system of equations in the explicit form

$$\sum_{n=0}^{N}\left\{\frac{1}{n+k} a_n''(z) + \frac{(2n)^2}{n+k-1}\frac{a_n(z)}{R_0^2}\right\} = 0$$

$$(k = 1, 2, \ldots, N),$$

$$\sum_{n=1}^{M}\left\{\frac{1}{n+k-1} b_n''(z) + \frac{(2n+1)^2}{n+k}\frac{b_n(z)}{R_0^2}\right\} = 0$$

$$(k = 1, 2, \ldots, M-1),$$

$$\sum_{n=0}^{N} a_n'(z) - \sum_{n=1}^{M} b_n'(z) = A,$$

$$\sum_{n=0}^{N} 2n a_n(z) + \sum_{n=1}^{M} (2n+1) b_n(z) = 0. \qquad (12)$$

Similar reasoning for the functions $\{a_n^*(z)\}$ and $\{b_n^*(z)\}$ gives us a similar system of equations:

$$\sum_{n=0}^{N}\left\{\frac{1}{n+k} a_n^{*''}(z) - \frac{4n}{n+k}\frac{a_n^{*'}(z)}{R(z)} + \left[\frac{(2n)^2}{n+k-1} + \frac{2n(2n+1)}{n+k}\right]\frac{a_n^*(z)}{R^2(z)}\right\} = 0$$

$$(k = 1, 2, \ldots, N),$$

$$\sum_{n=1}^{M}\left\{\frac{1}{n+k-1} b_n^{*''}(z) + \frac{2(2n+1)}{n+k-1}\frac{b_n^{*'}(z)}{R(z)} + \left[\frac{(2n+1)^2}{n+k} + \frac{2n(2n+1)}{n+k+1}\right]\frac{b_n^*(z)}{R^2(z)}\right\} = 0$$

$$(k = 1, 2, \ldots, M-1),$$

$$\sum_{n=0}^{N} a_n^{*'}(z) - \sum_{n=1}^{M} b_n^{*'}(z) = 0,$$

$$\sum_{n=1}^{N} 2n a_n^*(z) + \sum_{n=1}^{M} (2n+1) b_n^*(z) = 0. \qquad (13)$$

As usual we seek a solution of the homogeneous system (12) in the form

$$a_n(z) = \alpha_n \exp\left(\lambda \frac{z}{R_0}\right), \quad b_m(z) = \beta_m \exp\left(\lambda \frac{z}{R_0}\right)$$

$$(n = 0, 1, 2, \ldots, N;\ m = 1, 2, \ldots, M).$$

The characteristic equation of system (12) is a polynomial of degree $N + M - 1$ in λ^2. Consequently, in the general form this polynomial has $N + M - 1$ roots of λ^2, or $2(N + M - 1)$ roots of λ, which occur in pairs of $\pm\lambda$. There is in addition one zero root.

We consider the case in which all the roots $\pm\lambda_i$ $(i = 1, 2, \ldots, N + M - 1)$ are different. It is then possible to write $N + M - 1$ particular solutions of the homogeneous system (12) in the form

$$a_n = \alpha_n^i \operatorname{sh} \lambda_i \frac{z}{R_0}, \quad b_m = \beta_m^i \operatorname{sh} \lambda_i \frac{z}{R_0}$$

$$(i = 1, 2, \ldots, N + M - 1;$$

$$n = 0, 1, 2, \ldots, N;\ m = 1, 2, \ldots, M), \qquad (14)$$

where $\{\alpha_n^i, \beta_m^i\}$ is the eigenvector corresponding to the root λ_i^2.

It is not difficult to see that the nonhomogeneous system (12) may be satisfied if we take the expression Az instead of the constant α_0^0 (corresponding to the zero root). The general solution of system (12) may then be written as

$$a_0(z) = Az + \sum_{i=1}^{N} c_i \alpha_0^i \operatorname{sh} \lambda_i \frac{z}{R_0} + \sum_{i=N+1}^{N+M-1} c_i \beta_0^i \operatorname{sh} \lambda_i \frac{z}{R_0},$$

$$a_n(z) = \sum_{i=1}^{N} c_i \alpha_n^i \operatorname{sh} \lambda_i \frac{z}{R_0} + \sum_{i=N+1}^{N+M-1} c_i \beta_n^i \operatorname{sh} \lambda_i \frac{z}{R_0},$$

$$b_m(z) = \sum_{i=1}^{N} c_i \alpha_m^i \operatorname{sh} \lambda_i \frac{z}{R_0} + \sum_{i=N+1}^{N+M-1} c_i \beta_m^i \operatorname{sh} \lambda_i \frac{z}{R_0}$$

$$(n = 1, 2, \ldots, N;\ m = 1, 2, \ldots, M). \qquad (15)$$

Using Euler's method we now look for particular solutions of system (13) in the form

$$a_n^*(z) = \alpha_n^*(z-\gamma)^{\lambda}, \quad b_m^*(z) = \beta_m^*(z-\gamma)^{\lambda}$$

$$(n = 0,\ 1,\ 2,\ \ldots,\ N;\quad m = 1,\ 2,\ \ldots,\ M). \qquad (16)$$

The question of the behavior of the roots in the characteristic equation of system (13) is in general fairly complicated. We therefore consider the case in which there are equal numbers of positive and negative roots. All the positive roots must be rejected since they do not satisfy the boundary condition (5). The general solution of system (13) may then be written in the following form, similar to expression (15):

$$a_n^*(z) = \sum_{i=1}^{N} c_i^* \alpha_n^{*i} (z-\gamma)^{\lambda_i} + \sum_{i=N+1}^{N+M-1} c_i^* \beta_n^{*i} (z-\gamma)^{\lambda_i},$$

$$b_m^*(z) = \sum_{i=1}^{N} c_i^* \alpha_m^{*i} (z-\gamma)^{\lambda_i} + \sum_{i=N+1}^{N+M-1} c_i^* \beta_m^{*i} (z-\gamma)^{\lambda_i}$$

$$(n = 0,\ 1,\ 2,\ \ldots,\ N;\quad m = 1,\ 2,\ \ldots,\ M). \qquad (17)$$

The general solution of the problem is obtained by inserting expressions (15) and (17) into formulas (9). We determine the arbitrary constants c_i and c_i^* ($i = 1, 2, 3, \ldots, N + M - 1$) from the conditions for joining the solutions at the boundary of the regions where $z = l$. The conditions for joining the solutions at the boundary of regions (I, II) and (I*, II*) are obtained from the requirement that when the Laplace operator is applied to the solutions, the results should satisfy Eqs. (10) and (11) continuously on passing through the boundary of these regions. For this purpose we write the solution at any point in region (I, I*) in the form

$$\varphi(r,\ z) = \theta(z)\varphi_{\mathrm{I}}(r,\ z) + (1-\theta(z))\varphi_{\mathrm{I}}^*(r,\ z),$$

where

$$\theta(z) = \begin{cases} 1, & z < l, \\ 0, & z > l. \end{cases}$$

The second derivative with respect to the variable z is then written as

$$\frac{\partial^2 \varphi(r,\ z)}{\partial z^2} = \delta'(z-l)(\varphi_{\mathrm{I}} - \varphi_{\mathrm{I}}^*) + $$
$$+ 2\delta(z-l)(\varphi_{\mathrm{I}}' - \varphi_{\mathrm{I}}^{*\prime}) + \theta\frac{\partial^2 \varphi_{\mathrm{I}}}{\partial z^2} + (1-\theta)\frac{\partial^2 \varphi_{\mathrm{I}}^*}{\partial z^2},$$

where δ and δ' denote the Dirac delta function and its derivative.

Table 1

Eigenvectors for the Roots of the Characteristic Equation of System (12)

λ_i	α_0	α_1	β_1	β_2
0	1	0	0	0
±2.8056	1	−0.991895	−0.971633	0.979738
±0.67968	1	−0.109187	2.11785	−1.22703

Since the last two terms as well as the radial terms of the Laplace operator satisfy Eq. (10) at any point in space, the requirement that (10) should be satisfied at boundary points where $z = l$ leads to the following

Table 2

Eigenvectors for the Two Negative Roots of the Characteristic Equation of System (13)

λ_i	α_0^*	α_1^*	β_1^*	β_2^*
0	1	0	0	0
−2.1261	1	−0.456001	0.903996	−0.359997
−4.7633	1	−0.907429	−0.676004	0.768575

systems of equations for determining the constants c_i and c_i^*:

$$\int_0^{R_0} [\varphi_{\mathrm{I}}(r,\ l) - \varphi_{\mathrm{I}}^*(r,\ l)]\, r^{2k-1}\, dr = 0,$$

$$\int_0^{R_0} [\varphi_{\mathrm{I},z}'(r,\ z) - \varphi_{\mathrm{I},z}^{*\prime}(r,\ z)]_{z=l}\, r^{2k-1}\, dr = 0$$

$$(k = 1,\ 2,\ \ldots,\ N). \qquad (18)$$

We obtain similar systems of equations for the boundary of regions II and II*:

$$\int_{R_0}^{\infty} [\varphi_{\mathrm{II}}(r,\ l) - \varphi_{\mathrm{II}}^*(r,\ l)]\, r^{-2(k-1)}\, dr = 0,$$

$$\int_{R_0}^{\infty} [\varphi_{\mathrm{II},z}'(r,\ z) - \varphi_{\mathrm{II},z}^{*\prime}(r,\ z)]_{z=l}\, r^{-2(k-1)}\, dr = 0$$

$$(k = 1,\ 2,\ \ldots,\ M-1). \qquad (19)$$

Thus, a system of $2(N + M - 1)$ algebraic equations is obtained to determine $2(N + M - 1)$ arbitrary constants c_i and c_i^*.

By way of an example, let us solve the problem, confining ourselves to the second approximation of the method described above, i.e., we set $N = 1$ and $M = 2$.

In this case the characteristic equation of system (12) is

$$\lambda(33\lambda^4 - 275\lambda^2 + 120) = 0,$$

which has the roots $\lambda_0 = 0$, $\lambda_{1,2} = \pm 2.8056$ and $\lambda_{3,4} = \pm 0.67968$. The eigenvectors for each of these roots are given in Table 1.

The characteristic equation of system (13) has the form

$$\lambda(33\lambda^4 + 132\lambda^3 - 302\lambda^2 - 823\lambda + 210) = 0.$$

In addition to the zero root this equation has two positive and two negative roots. We are not interested in the positive roots. The negative roots are $\lambda_1 = -2.1261$, and $\lambda_2 = -4.7633$.

The eigenvectors corresponding to these two roots are given in Table 2.

The values of the constants c_i and c_i^* are

$$c_1 = -0.000886,\quad c_2 = -0.1233,$$
$$c_0^* = 1,\quad c_1^* = -0.3739,\quad c_2^* = 0.02843.$$

for the case in which $l = 2R_0$, $R_0 = 1$, $A = 1$.

In this approximation the calculated field strength on the axis of the coil at the center ($z = 0$) is equal to 0.9137, and at the edge of the coil ($z = l$) it is equal to 0.4862. The corresponding values found by solving the problem exactly are 0.8945 and 0.4848. The approximate value at the center of the coil is higher by 2%, and at the edge by 0.3%.

REFERENCES

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